Scalars, Vectors, and Phasors

A **scalar** is a physical quantity that has a **magnitude** (number and dimension) only. Examples: mass, DC voltage, DC current, charge, flux.

A **vector** is a physical quantity that has a **magnitude** and a **direction**. Examples: force, torque, magnetic field.

A **phasor** is a physical quantity that has an **amplitude** (or magnitude) and a **phase**. Phasors are particularly popular among electrical engineers and are used to describe the phase relationships between currents, voltages, and other **quantities in an AC circuit**.

Example: The applied voltage $v(t)$ given by the following expression has an amplitude v_0 and a phase j :

$$v(t) = v_0 \sin(wt + j)$$

Since a complex number $z = |z| e^{\mathrm{i} f}$ has a magnitude $|z|$ and a phase j , it is convenient to describe phasors with complex numbers.

Moivre's theorem states that $e^{\mathrm{i} f} = \cos f + \mathrm{i} \sin f$, therefore the voltage in the example above is

$$v(t) = v_0 \sin(wt + j) = v_0 \operatorname{Im} e^{\mathrm{i}(wt + f)} = \operatorname{Im}\left[v_0 e^{\mathrm{i} wt} e^{\mathrm{i} f}\right].$$

The **time-independent part** $v_0 e^{\mathrm{i} f}$ of this complex expression is the **phasor**.

See your calculus book (appendix) for more information about complex numbers.

Phasors in an AC circuit

Let us consider an AC circuit containing a resistor R, an inductor L, and a capacitor C. The **loop rule** states that

$$v - v_R - v_L - v_C = 0.$$

Since this is an AC circuit, we not only have to consider the amplitudes of these voltages, but also their **phases**. We therefore describe these voltages using **phasors**.

$v = v_0 e^{\mathrm{i} f}$	applied voltage	$v = v_0 e^{\mathrm{i} f}$
$v_R = v_{0R} e^{\mathrm{i} f_R}$	voltage across R	$v_R = v_{0R}$
$v_L = v_{0L} e^{\mathrm{i} f_L}$	voltage across L	$v_L = \mathrm{i} v_{0L}$
$v_C = v_{0C} e^{\mathrm{i} f_C}$	voltage across C	$v_C = -\mathrm{i} v_{0C}$

Since current and voltage are in phase across a resistor and the current has a phase angle of zero by definition, $f_R = 0$.

v_L leads i, therefore $f_L = p/2$. Thus, $e^{i f_L} = \mathrm{i}$.

v_C lags i, therefore $f_C = -p/2$. Thus, $e^{i f_C} = -\mathrm{i}$.

Our next goal is to solve for the **phase angle** f in Kirchhoff's loop rule in terms of R, C, and L. We also want to find the **impedance** Z such that $v_0 = i_0 Z$ or $V = IZ$.

Let us summarize: The angle f is the phase between the current and the voltage. The impedance Z is the ratio of voltage to current.

Impedance

The loop rule $v - v_R - v_L - v_C = 0$ implies the following phasor equation:

$$v_0 e^{\mathrm{i} f} = v_{0R} + \mathrm{i}\left(v_{0L} - v_{0C}\right)$$

By taking the magnitude on both sides, we see that

$$v_0 = \sqrt{v_{0R}^2 + \left(v_{0L} - v_{0C}\right)^2}$$

Using Ohm's law $v_{0R} = i_0 R$ and the definition of the reactance for a capacitor $v_{0C} = i_0 X_C$ and an inductor $v_{0L} = i_0 X_L$, this becomes

$$v_0 = i_0 \sqrt{R^2 + \left(X_L - X_C\right)^2}$$

We define the **impedance** as $Z = \sqrt{R^2 + \left(X_L - X_C\right)^2}$. Then we get back our usual equations $v_0 = i_0 Z$ and $V = IZ$ for the relationships between current and voltage.

The phase angle

The phase angle f between current and voltage is

$$\tan f = \frac{v_{0L} - v_{0C}}{v_{0R}} = \frac{X_L - X_C}{R}$$

Power in AC cirucuits

The instantaneous power is $p(t) = i(t)v(t)$. Therefore,

$$p = i_0 \sin(wt) v_0 \cos(wt + j)$$

$$p = i_0 v_0 \sin(wt)[\sin(wt)\cos f + \cos(wt)\sin j]$$

When we integrate over time in order to get the average power, the quadratic term gives us a contribution $\left[\sin^2(wt)\right]_{\mathrm{av}} = \frac{1}{2}$, but the mixed term $[\sin(wt)\cos(wt)]_{\mathrm{av}} = 0$ vanishes.

The average power is

$$P = p_{\mathrm{av}} = \tfrac{1}{2} i_0 v_0 \cos j = IV \cos f = I^2 R.$$

As expected, power is only dissipated in the resistor.